

A NOTE ON ROUGH I -CONVERGENCE OF DOUBLE SEQUENCES

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ABSTRACT. In this paper we study some basic properties of rough I -convergent double sequences in the line of Dündar [8]. We also study the set of all rough I -limits of a double sequence and relation between boundedness and rough I -convergence of a double sequence.

Key words and phrases : Double sequence, ideal, rough I -convergence, rough I -limit.

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1. Introduction:

The concept of I -convergence of double sequences was introduced by Balcerzak et. al. [2]. The notion of I -convergence of a double sequence, which is based on the structure of the ideal I of subsets of $\mathbb{N} \times \mathbb{N}$, where \mathbb{N} is the set of all natural numbers, is a natural generalization of the notion of convergence of a double sequence in Pringsheim's sense [17] as well as the notion of statistical convergence of a double sequence [14].

A lot of work on I -convergence of double sequences can be found in ([3], [4], [5], [7] etc.) and many others.

The concept of rough I -convergence of single sequences was introduced by Pal et. al. [15] which is a generalization of the earlier concepts namely rough convergence [16] and rough statistical convergence [1] of single sequences. Recently rough statistical convergence of double sequences has been introduced by Malik and Maity [13] as a generalization of rough convergence of double sequences [12] and investigated some basic properties of this type of convergence and also studied relation between the set of statistical cluster points and the set of rough limit points of a double sequence. Recently the notion of rough I -convergence

for double sequences has been introduced by Dündar [8]. In this paper we investigate some basic properties of rough I -convergence of double sequences in finite dimensional normed linear spaces which are not done earlier. We study the set of rough I -limits of a double sequence and also the relation between boundedness and rough I -convergence of a double sequence.

2. Basic Definitions and Notations

Throughout the paper \mathbb{N} denotes the set of all positive integers and \mathbb{R} denotes the set of all real numbers.

Definition 2.1 (12). Let $x = \{x_{jk}\}_{j,k \in \mathbb{N}}$ be a double sequence in a normed linear space $(X, \|\cdot\|)$ and r be a non-negative real number. x is said to be r -convergent to $\xi \in X$, denoted by $x \xrightarrow{r} \xi$, if for any $\epsilon > 0$ there exists $N_\epsilon \in \mathbb{N}$ such that for all $j, k \geq N_\epsilon$ we have

$$\|x_{jk} - \xi\| < r + \epsilon.$$

In this case ξ is called an r -limit of x .

It is clear that rough limit of x is not necessarily unique (for $r > 0$). So we consider r -limit set of x which is denoted by LIM_x^r and is defined by $LIM_x^r = \{\xi \in X : x \xrightarrow{r} \xi\}$. x is said to be r -convergent if $LIM_x^r \neq \emptyset$ and r is called a rough convergence degree of x .

We recall that a subset K of $\mathbb{N} \times \mathbb{N}$ is said to have natural density $d(K)$ if

$$d(K) = \lim_{n \rightarrow \infty} \frac{K(n, m)}{n \cdot m},$$

where $K(n, m) = |\{(j, k) \in \mathbb{N} \times \mathbb{N} : j \leq n, k \leq m\}|$.

Definition 2.2 (13). Let $x = \{x_{jk}\}_{j,k \in \mathbb{N}}$ be a double sequence in a normed linear space $(X, \|\cdot\|)$ and r be a non-negative real number. x is said to be r -statistically convergent to ξ , denoted by $x \xrightarrow{r-st_2} \xi$, if for any $\epsilon > 0$ we have $d(A(\epsilon)) = 0$, where $A(\epsilon) = \{(j, k) \in \mathbb{N} \times \mathbb{N} : \|x_{jk} - \xi\| \geq r + \epsilon\}$. In this case ξ is called r -statistical limit of x .

Clearly for $r = 0$ from Definition 2.1 we get Pringsheim convergence of double sequences and from Definition 2.2 we get ordinary statistical convergence of double sequences.

Definition 2.3. A class I of subsets of a nonempty set X is said to be an ideal in X provided

- (i) $\phi \in I$.
 - (ii) $A, B \in I$ implies $A \cup B \in I$.
 - (iii) $A \in I, B \subset A$ implies $B \in I$.
- I is called a nontrivial ideal if $X \notin I$.

Definition 2.4. A non empty class F of subsets of a nonempty set X is said to be a filter in X provided

- (i) $\phi \notin F$.
- (ii) $A, B \in F$ implies $A \cap B \in F$.
- (iii) $A \in F, A \subset B$ implies $B \in F$.

If I is a nontrivial ideal in X , $X \neq \phi$, then the class

$$F(I) = \{M \subset X : M = X \setminus A \text{ for some } A \in I\}$$

is a filter on X , called the filter associated with I .

Definition 2.5 (4). A nontrivial ideal I in X is called admissible if $\{x\} \in I$ for each $x \in X$.

Definition 2.6 (4). A nontrivial ideal I on $\mathbb{N} \times \mathbb{N}$ is called strongly admissible if $\{i\} \times \mathbb{N}$ and $\mathbb{N} \times \{i\}$ belong to I for each $i \in \mathbb{N}$.

Clearly every strongly admissible ideal is admissible. Throughout the paper we take I as a strongly admissible ideal in $\mathbb{N} \times \mathbb{N}$.

Definition 2.7 (8). Let $x = \{x_{jk}\}_{j,k \in \mathbb{N}}$ be a double sequence in a normed linear space $(X, \|\cdot\|)$ and r be a non negative real number. Then x is said to be rough ideal convergent or rI -convergent to ξ , denoted by $x \xrightarrow{rI} \xi$, if for any $\varepsilon > 0$ we have $\{(j, k) \in \mathbb{N} \times \mathbb{N} : \|x_{jk} - \xi\| \geq r + \varepsilon\} \in I$. In this case ξ is called rI -limit of x and x is called rough I -convergent to ξ with r as roughness degree.

Throughout this paper x denotes the double sequence $\{x_{jk}\}_{j,k \in \mathbb{N}}$ in a normed linear space $(X, \|\cdot\|)$ and r denotes a non negative real number.

For $r = 0$ we get the usual I -convergence of double sequences. But our main interest is on the case where $r > 0$. Because it may happen that a double sequence $x = \{x_{jk}\}_{j,k \in \mathbb{N}}$ is not I -convergent in usual sense but there exists a double sequence $y = \{y_{jk}\}_{j,k \in \mathbb{N}}$ which is I -convergent in usual sense and $\|x_{jk} - y_{jk}\| \leq r$ for all $(j, k) \in \mathbb{N} \times \mathbb{N}$ (or $\{(j, k) \in \mathbb{N} \times \mathbb{N} : \|x_{jk} - y_{jk}\| > r\} \in I$) for some $r > 0$. Then x is rI -convergent.

From the definition it is clear that rI -limit of x is not necessarily unique (for $r > 0$). So we consider rI -limit set of x , which is denoted by $I - LIM_x^r = \{\xi \in X : x \xrightarrow{rI} \xi\}$. x is said to be rI -convergent if $I - LIM_x^r \neq \emptyset$ and r is called a rough I -convergence degree of x .

Definition 2.8. A double sequence x in X is said to be bounded if there exists a positive real number M such that $\|x_{jk}\| < M$ for all $(j, k) \in \mathbb{N} \times \mathbb{N}$.

Definition 2.9. A double sequence x in X is said to be I -bounded if there exists a positive real number M such that $\{(j, k) \in \mathbb{N} \times \mathbb{N} : \|x_{jk}\| \geq M\} \in I$.

Definition 2.10 (5). A point $\xi \in X$ is said to be an I -cluster point of a double sequence $x = \{x_{jk}\}_{j,k \in \mathbb{N}}$ if and only if for each $\varepsilon > 0$ the set $\{(j, k) \in \mathbb{N} \times \mathbb{N} : \|x_{jk} - \xi\| < \varepsilon\} \notin I$. We denote the set of all I -cluster points of x by $I(\Gamma_x)$.

Theorem 2.1 (5). *An I -bounded double sequence $x = \{x_{jk}\}_{j,k \in \mathbb{N}}$ of real numbers is I -convergent if and only if $I - \limsup x = I - \liminf x$.*

Theorem 2.2 (5). *Let $x = \{x_{jk}\}_{j,k \in \mathbb{N}}$ be a bounded double sequence of real numbers, then*

- (i) $I - \limsup x = \max I(\Gamma_x)$,
- (ii) $I - \liminf x = \min I(\Gamma_x)$.

The above result is also true for I -bounded double sequences. So it can be stated as follows.

Theorem 2.3. *Let $x = \{x_{jk}\}_{j,k \in \mathbb{N}}$ be an I -bounded double sequence of real numbers, then*

- (i) $I - \limsup x = \max I(\Gamma_x)$,
- (ii) $I - \liminf x = \min I(\Gamma_x)$.

Theorem 2.4 (8). *For a double sequence $x = \{x_{jk}\}_{j,k \in \mathbb{N}}$ in a normed linear space $(X, \|\cdot\|)$ we have $\text{diam}(I - LIM_x^r) \leq 2r$. In particular if $x \xrightarrow{I} \xi$, then $I - LIM_x^r = \overline{B_r}(\xi) = \{y \in X : \|y - \xi\| \leq r\}$ and so $\text{diam}(I - LIM_x^r) = 2r$.*

Note 2.1. *When $r=0$, then $\text{diam}(I - LIM_x^r) = 0$. Therefore $I - LIM_x^r$ is either \emptyset or singleton. This implies the uniqueness of limit of I -convergent double sequence.*

Theorem 2.5 (8). *Let $x = \{x_{jk}\}_{j,k \in \mathbb{N}}$ be a double sequence in X and $c \in I(\Gamma_x)$. Then $\|\xi - c\| \leq r$ for all $\xi \in I - LIM_x^r$ i.e. $I - LIM_x^r \subset \overline{B_r}(c)$.*

We now consider an example of a double sequence which is rough I -convergent but not rough convergent.

Example 2.1. *We consider the ideal $I_d = \{A \subset \mathbb{N} \times \mathbb{N} : d(A) = 0\}$. Let $x = \{x_{jk}\}_{j,k \in \mathbb{N}}$ be a double sequence in the normed linear space $(\mathbb{R}, \|\cdot\|)$ defined by*

$$\begin{aligned} x_{jk} &= 2jk, \text{ if } j \text{ and } k \text{ are squares,} \\ &= (-1)^{j+k}, \text{ otherwise.} \end{aligned}$$

Then

$$I_d - LIM_x^r = \begin{cases} \emptyset & ; \text{ if } r < 1 \\ [1-r, r-1] & ; \text{ if } r \geq 1 \end{cases}$$

and $LIM_x^r = \emptyset$ for all $r \geq 0$.

From the above example we see that $I - LIM_x^r \neq \emptyset$ does not imply $LIM_x^r \neq \emptyset$. But $LIM_x^r \neq \emptyset$ always implies that $I - LIM_x^r \neq \emptyset$.

3. Main Results

We first establish a relation between boundedness and rough I -convergence of double sequences.

Theorem 3.1. *If a double sequence $x = \{x_{jk}\}$ is bounded, then there exists $r \geq 0$ such that $I - LIM_x^r \neq \emptyset$.*

Proof. The proof is similar to the proof of Theorem 3.2 [13], so is omitted. \square

Note 3.1. *Taking $I = \{A \in \mathbb{N} \times \mathbb{N} : d(A) = 0\}$, from Note 3.2 [13] we see that the converse of Theorem 3.1 is not true.*

We now show that the converse of Theorem 3.1 is true if the double sequence x is I -bounded.

Theorem 3.2. *A double sequence x is I -bounded if and only if there exists $r \geq 0$ such that $I - LIM_x^r \neq \emptyset$.*

Proof. Let x be an I -bounded double sequence. Then there exists a positive real number M such that $A = \{(j, k) \in \mathbb{N} \times \mathbb{N} : \|x_{jk}\| \geq M\} \in I$. Let $r' = \sup\{\|x_{jk}\| : (j, k) \in \mathbb{N} \times \mathbb{N} \setminus A\}$. Then $0 \in I - LIM_x^{r'}$ and so $I - LIM_x^{r'} \neq \emptyset$.

Conversely, let $I - LIM_x^r \neq \emptyset$ for some $r \geq 0$. Let $\xi \in I - LIM_x^r$. Take $\varepsilon = 1$. Then $B = \{(j, k) \in \mathbb{N} \times \mathbb{N} : \|x_{jk} - \xi\| \geq 1 + r\} \in I$. Now $\{(j, k) \in \mathbb{N} \times \mathbb{N} : \|x_{jk}\| \geq 1 + r + \|\xi\|\} \subset B$ and so $\{(j, k) \in \mathbb{N} \times \mathbb{N} : \|x_{jk}\| \geq 1 + r + \|\xi\|\} \in I$. This shows that x is I -bounded. \square

Next we present an alternative proof of Theorem 2.4 [8] which gives a topological property of the rI -limit set of a double sequence.

Theorem 3.3. *For all $r \geq 0$, the rI -limit set $I - LIM_x^r$, of a double sequence $x = \{x_{jk}\}_{j,k \in \mathbb{N}}$ is closed.*

Proof. Let ξ be a limit point of $I - LIM_x^r$. Then for any $\varepsilon > 0$, $B_{\frac{\varepsilon}{2}}(\xi) \cap I - LIM_x^r \neq \emptyset$. Let $\alpha \in B_{\frac{\varepsilon}{2}}(\xi) \cap I - LIM_x^r$. Since $\alpha \in I - LIM_x^r$ so $A(\frac{\varepsilon}{2}) = \{(j, k) \in \mathbb{N} \times \mathbb{N} : \|x_{jk} - \alpha\| \geq r + \frac{\varepsilon}{2}\} \in I$. Let $B(\varepsilon) = \{(j, k) \in \mathbb{N} \times \mathbb{N} : \|x_{jk} - \xi\| \geq r + \varepsilon\}$. Now $(j, k) \notin A(\frac{\varepsilon}{2})$ implies $(j, k) \notin B(\varepsilon)$. Thus $(j, k) \in B(\varepsilon)$ implies $(j, k) \in A(\frac{\varepsilon}{2})$. This implies $B(\varepsilon) \subset A(\frac{\varepsilon}{2})$ and so $B(\varepsilon) = \{(j, k) \in \mathbb{N} \times \mathbb{N} : \|x_{jk} - \xi\| \geq r + \varepsilon\} \in I$. Therefore $\xi \in I - LIM_x^r$. Hence $I - LIM_x^r$ is a closed set in X . \square

Theorem 3.4. *Let $x = \{x_{jk}\}_{j,k \in \mathbb{N}}$ be a double sequence in X . Then x is I -convergent to ξ if and only if $I - LIM_x^r = \overline{B_r}(\xi)$.*

Proof. It directly follows from Theorem 2.4 that if x is I -convergent to ξ , then $I - LIM_x^r = \overline{B_r}(\xi)$.

Conversely, let $I - LIM_x^r = \overline{B_r}(\xi)$. We have to show that x is I -convergent to ξ , i.e. for all $a > 0$, $A(a) = \{(j, k) \in \mathbb{N} \times \mathbb{N} : \|x_{jk} - \xi\| \geq a\} \in I$. Now fixed

$a > 0$. Let us choose $r > 0$ and $\varepsilon > 0$ such that $r + \varepsilon < a$. For $\xi \in I - LIM_x^r$, $\{(j, k) \in \mathbb{N} \times \mathbb{N} : \|x_{jk} - \xi\| \geq r + \varepsilon\} \in I$. Since $\{(j, k) \in \mathbb{N} \times \mathbb{N} : \|x_{jk} - \xi\| \geq a\} \subset \{(j, k) \in \mathbb{N} \times \mathbb{N} : \|x_{jk} - \xi\| \geq r + \varepsilon\}$. So $\{(j, k) \in \mathbb{N} \times \mathbb{N} : \|x_{jk} - \xi\| \geq a\} \in I$. Hence x is I -convergent to ξ . \square

Theorem 3.5. *Let $(\mathbb{R}, \|\cdot\|)$ be a strictly convex space and $x = \{x_{jk}\}_{j,k \in \mathbb{N}}$ be double sequence in \mathbb{R} . For any $r > 0$, let $y_1, y_2 \in I - LIM_x^r$ with $\|y_1 - y_2\| = 2r$. Then x is I -convergent to $\frac{1}{2}(y_1 + y_2)$.*

Proof. Let y_3 be an arbitrary I -cluster point of x . Now since $y_1, y_2 \in I - LIM_x^r$, so by Theorem 2.5 we have

$$\|y_1 - y_3\| \leq r \text{ and } \|y_2 - y_3\| \leq r.$$

Then $2r = \|y_1 - y_2\| \leq \|y_1 - y_3\| + \|y_3 - y_2\| \leq 2r$. Therefore $\|y_1 - y_3\| = \|y_2 - y_3\| = r$. Now

$$\frac{1}{2}(y_1 - y_2) = \frac{1}{2}[(y_3 - y_1) + (y_2 - y_3)]. \quad (1)$$

Since $\|y_1 - y_2\| = 2r$, so $\frac{1}{2}\|y_2 - y_1\| = r$. Again since the space is strictly convex, so by (1) we get $\frac{1}{2}(y_2 - y_1) = y_3 - y_1 = y_2 - y_3$. Thus y_3 is the unique I -cluster point of the double sequence x . Again by the given condition $I - LIM_x^r \neq \emptyset$, so by Theorem 3.2 x is I -bounded. Since y_3 is the unique I -cluster point of the I -bounded double sequence x , so by Theorem 2.1 and Theorem 2.3 x is I -convergent to $y_3 = \frac{1}{2}(y_1 + y_2)$. \square

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REFERENCES

- [1] S. Aytar: Rough statistical coverage, Numer. Funct. Anal. And Optimiz., 29(3)(2008), 291-303.
- [2] M. Balcerzak, K. Dems: Some types of convergence and related Baire systems, Real. Anal. Exchange, 30(1)(2004/2005), 267-276.
- [3] P. Das, P. Malik: On the statistical and I -variation of double sequences, Real Anal. Exchange, 33(2007), 351-364.
- [4] P. Das, P. Kostyrko, W. Wilczyński, P. Malik: I and I^* -convergence of double sequences, Math. Slovaca, 58(2008), 605-620.
- [5] P. Das, P. Malik: On extremal I -limit points of double sequences, Tatra Mt. Math. Publ, 40 (2008), 91-102.
- [6] K. Demirci: I -limit superior and limit inferior, Math. Commun., 6(2)(2001), 165-172.
- [7] K. Dems: On I -Cauchy sequences, Real Anal. Exchange, 30(1)(2004/2005), 123-128.
- [8] E. Dündar: On rough I_2 -convergence of double sequences, Numer. Funct. Anal. And Optimiz., DOI:10.1080/01630563.2015.1136326.

- [9] P. Kostyrko, T. Šalát, W. Wilczyński: I -convergence, Real Anal. Exchange, 26(2)(2000/2001), 669-685.
- [10] P. Kostyrko, M. Macaz, T. Šalát, M. Szeziak: I -convergence and external I -limit points, Math. Slovaca, 55(4)(2005), 443-454.
- [11] B.K. Lahiri, P. Das: I and I^* convergence in topological spaces, Math. Bohemica, 130(2)(2005), 153-160.
- [12] P. Malik and M. Maity: On rough convergence of double sequence in normed linear spaces, Bull. Allah. Math. Soc., 28(1)(2013), 89-99.
- [13] P. Malik and M. Maity: On rough statistical convergence of double sequences in normed linear spaces, Afr. Mat., (2016) 27: 141-148.
- [14] M. Mursaleen, O.H.H. Edely : Statistical Convergence of double sequences, J. Math. Anal. Appl, 288 (2003), 223-231.
- [15] S.K. Pal, D. Chandra, S. Dutta: Rough ideal convergence, Hacettepe J. of Math. and Stat., 42(6)(2013), 633-640.
- [16] H.X. Phu: Rough convergence in normed linear spaces, Numer. Funct. Anal. And Optimiz., 22(2001), 201-224.
- [17] A. Pringsheim: Zur theortie der Gamma-Functionen, Math. Annalen, 31 (1888), 455-481.
- [18] A. Zygmund: Trigonometric Series, Cambridge University Press, Cambridge, UK, 1979.